1.7 Linear Independence

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"We've begun to notice a deep connection between the behavior of the solutions of $A\vec{x}=\vec{b}$ and the reduced echelon form of A: e.g. - if all rows of A have pivots, the columns of A span $\mathbb{R}^n$ and this equation is solvable for all $\vec{b}$.

or

- if all columns of A have pivots, the homogeneous equation $A\vec{x}=\vec{0}$ has only the trivial solution and $A\vec{x}=\vec{b}$ has exactly one solution, if any.

The idea of having enough of the right vectors to solve any system will appear later in the notion of basis and will connect these two ideas (ergo, pivots in all rows and all columns of A), for now we give a definition to generalize the second case (A has pivots in each column.)"

Def: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution (i.e. $[\vec{v}_1 \cdots \vec{v}_p]\vec{x}=\vec{0}$ has only the trivial sol'n).

If there are $c_1, \dots, c_p$, not all 0, s.t.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = \vec{0} \longleftarrow$$

we say $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent.

In this case, we call this equation a linear dependence relation for $\vec{v}_1, \dots, \vec{v}_p$

Ex| Determine if $\vec{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$ are linearly independent.

Nope

To determine this, need see if there are nontrivial x_1, x_2, x_3 s.t.

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

$$\Updownarrow$$

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} + x_3\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \nearrow \begin{bmatrix}1&4&2\\2&5&1\\3&6&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

so we row reduce $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{0}]$

$$\begin{bmatrix}1&4&2&0\\2&5&1&0\\3&6&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&0\end{bmatrix}$$

$$x_1\begin{bmatrix}2\\3\end{bmatrix}+x_2\begin{bmatrix}5\\6\end{bmatrix}+x_3\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

so we row reduce $\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0}\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

Notice x_3 is free, so there **is** a nontrivial solution: $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ are linearly dependent.

Specifically $x_1 = 2x_3$, $x_2 = -x_3$, $x_3 \neq 0$ if $x_3 = 1 \Rightarrow x_1 = 2$, $x_2 = -1$ and $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$. ✓

Ex/ Determine if the columns of $A = \begin{bmatrix}0 & 1 & 4\\ 1 & 2 & -1\\ 5 & 8 & 0\end{bmatrix}$ are linearly independent.

Similarly $\Rightarrow$ $\begin{bmatrix}0 & 1 & 4 & 0\\ 1 & 2 & -1 & 0\\ 5 & 8 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 2 & -1 & 0\\ 0 & 1 & 4 & 0\\ 0 & 0 & 13 & 0\end{bmatrix}$ so **yes**, only trivial solution $\Rightarrow$ $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{0}$ iff $x_1 = x_2 = x_3 = 0$.

$\hookrightarrow$ no free variable

Geometry of linear independence

Notice: A set of vectors $\vec{v}_1,\dots,\vec{v}_p$ is linearly dependent if:

- for some i, $\vec{v}_i = \vec{0}$. or
- for some j, $\vec{v}_j \neq \vec{0}$ but is a linear combination of the preceding vectors $\vec{v}_1,\dots,\vec{v}_{j-1}$.

Why? Notice if $\vec{v}_1,\dots,\vec{v}_{p-1}$ all not $\vec{0}$ but $\vec{v}_p = \vec{0}$ then

$c_1 = c_2 = \dots = c_{p-1} = 0$, $c_p \neq 0$, not all zero. ✓ $\Rightarrow$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 7\vec{v}_p = 7\cdot\vec{0} = \vec{0}.$$

$\rightarrow$ linear dependence relation

For the second case, if $\vec{v}_j \neq \vec{0}$ is a linear combination of $\vec{v}_1,\dots,\vec{v}_{j-1}$ then there are $c_1,\dots,c_{j-1}$ (not all zero, why?) s.t. $c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1} = \vec{v}_j$

$$\Rightarrow c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j = \vec{0}$$

So

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_{j-1}\vec{v}_{j-1} + \underbrace{(-1)}_{c_j}\vec{v}_j + \underbrace{0}_{c_{j+1}}\vec{v}_{j+1} + \dots + \underbrace{0}_{c_{p-1}}\vec{v}_{p-1} + \underbrace{0}_{c_p}\vec{v}_p = \vec{0}$$

is a linear dependence relation for $\vec{v}_1,\dots,\vec{v}_p$ ($c_1,\dots,c_p$ not all zero).

"Okay, so if the zero vector is in the collection it's automatically linearly dependent, otherwise, it's dependent iff one of the vectors is a linear combination of the others, this yields a geometric understanding of linear dependence (and therefore linear independence)."

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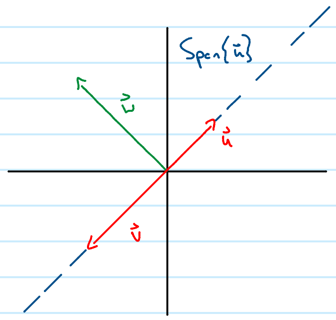
Ex Two vectors are linearly dependent if they are scalar multiples of one another: let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.



$-2\vec{v} = \vec{u} \Rightarrow \vec{u} - 2\vec{v} = \vec{0}$ is a lin. dep. relation

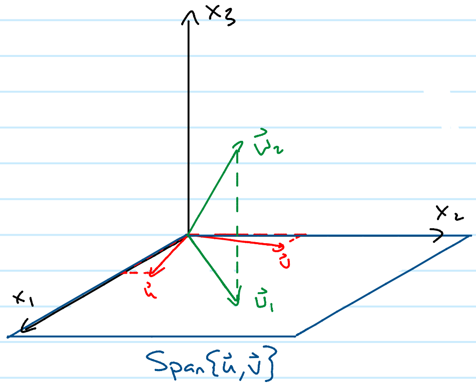
Notice $\vec{w}$ is not a scalar multiple of either vector so

$\{\vec{u}\}$, $\{\vec{v}\}$, $\{\vec{w}\}$, $\{\vec{u},\vec{w}\}$, $\{\vec{v},\vec{w}\}$ are linearly independent

$\{\vec{u},\vec{v}\}$, $\{\vec{u},\vec{v},\vec{w}\}$ are linearly dependent.

Ex Let $\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$. Then $\{\vec{u},\vec{v}\}$ is linearly independent.

$(\vec{u} \notin \operatorname{Span}\{\vec{v}\}$ and $\vec{v} \notin \operatorname{Span}\{\vec{u}\})$



Note: For any $\vec{w}$, $\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent so long as $\vec{w}$ isn't a linear combination of $\vec{u}$ and $\vec{v}$

i.e. if $\vec{w} \notin \operatorname{Span}\{\vec{u},\vec{v}\}$.

$\{\vec{u},\vec{v},\vec{w}_1\}$ is linearly dependent.

$\{u,v,\vec{w}_2\}$ is linearly independent.

“One final fact, linear indepence relies on a homogeneous equation having only the trivial solution: from this we see”

Fact: If the collection $\{\vec{v}_1,\dots,\vec{v}_p\}$ of vectors in $\mathbb{R}^n$ has more vectors than the number of entries in each vector $(p>n)$ then $\{\vec{v}_1,\dots,\vec{v}_p\}$ is linearly dependent.

Why? $[\vec{v}_1 \cdots \vec{v}_p]\vec{x} = \vec{0}$ has a nontrivial solution if there is a free variable: $[\vec{v}_1 \cdots \vec{v}_p] = \underbrace{\left.\begin{bmatrix} - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \end{bmatrix}\right\}n}_{p}$

this matrix can only have three pivots, but has many more this matrix can only have three pivots, but has many more columns. Ergo, some variable in the corresponding homogeneous system is free, and thus there is a nontrivial solution which in turns yields a linear dependence relation among the vectors $\vec{v}_1, \dots, \vec{v}_p$.